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THE INITIAL STAGE OF COLLAPSE OF SYMMETRIC GASEOUS PRISMS†

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An algorithm is developed for calculating the parameters of a shock-free compressible gas of symmetric tetrahedral gas prisms according to a given law of motion of a section of a piston using characteristic series. It is shown that by taking account of symmetry one can determine uniquely the parameters of the gas flow and the piston shape in a compressible prism. Three subdomains are distinguished in the domain of the perturbed gas flow, in each of which a velocity potential is constructed in the form of a characteristic series. In one subdomain, the solution is defined by the specified law of motion of the piston and, in the other two subdomains the solutions are defined by the conditions that no gas flows across the plane of symmetry and the continuity of its parameters at the boundaries along which the subdomains are joined, which are characteristics. An example of the calculation of a solution is given which, in a part of the domain, is identical to a known, self-similar exact solution when the adiabatic exponent and the magnitude of the dihedral angle of the prism are matched. An exact expression is obtained for the velocity potential and the shape of the wall of the gas prism being compressed.

The process of unbounded, shock-free compression of gas prisms, tetrahedra, and cone shaped bodies of special forms has been constructed in [1–5]. Exact solutions have been obtained for certain shapes. It has been shown that much less energy is required to achieve high gas densities during the compression of such structures than in the case of one-dimensional spherical compressions which are used to initiate laser thermonuclear synthesis.

It has been found [4, 5] when investigating the possibility of realising these processes during compression using different physical fields that closed structures,‡ in which there are no fixed impermeable walls, such as symmetric prisms with two planes of symmetry, are preferable. Below, we consider the problem of the shock-free compression of such prisms when the law of motion of a certain part of the piston is known and the solution of the problem satisfies the general non-linear equation for the velocity potential. An algorithm, developed but not published by A. F. Sidorov, for the case when the velocity potential of the gas satisfies a wave equation was taken as the basis for constructing a solution. Moreover, he showed that the use of this equation does not enable one in principle to describe unbounded cumulation of the density and energy of the gas.

1. FORMULATION OF THE PROBLEM AND METHOD OF SOLUTION

Suppose that, at the initial instant of time $t = 0$, a perturbed ideal gas with an equation of state $p = a^2\rho^\gamma$ (p is the pressure, ρ is the density and γ is the adiabatic exponent) occupies a certain volume A_1CDB_1 of a tetrahedral, infinite prism which is symmetrical about the planes $x = 0, y = 0$. One quadrant of the cross-section of this prism is shown in Fig. 1: α is the angle of inclination of one edge of the prism to the plane of symmetry $y = 0$. The gas prism is compressed symmetrically and in shock-free manner in accordance with a specified law of motion of the piston CD in the segment AB. In the case of an arbitrarily specified law, compression can be achieved up to a certain instant of time $t = t_k$. The time for a sound wave A_1B_1 to travel the distance $|OH| = 1$ corresponds to $t = 1$. In the gas at rest, the speed of sound $c = 1$.

In the case of shock-free compression, the perturbed gas flow is potential and satisfies the non-linear equation for the velocity potential $\Phi(x, y, t)$ [6]

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‡GAO, Ya. M., Computational realization of shock-free compression and thermonuclear combustion of two-dimensional (cylindrical and axially symmetric) configurations of a DT-gas. Preprint No, 21. M. V. Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Moscow, 2000.

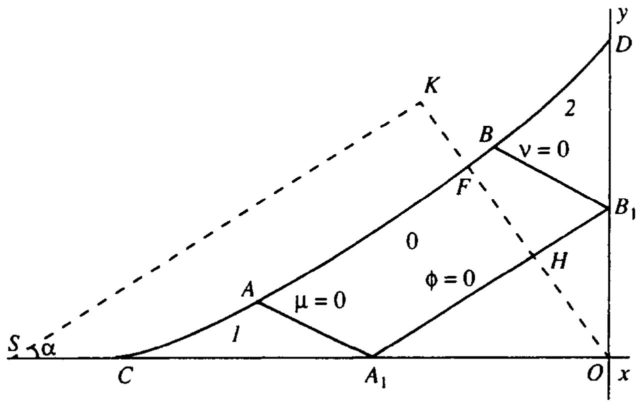

Fig. 1

$$\Phi_{tt}-(c^2-\Phi_x^2)\Phi_{xx}-(c^2-\Phi_y^2)\Phi_{yy}+2\Phi_x\Phi_{xt}+2\Phi_y\Phi_{yt}+2\Phi_x\Phi_y\Phi_{xy}=0 \tag{1.1}$$

$$c^2=1-(\gamma-1)[\Phi_t-\tfrac{1}{2}(\Phi_x^2+\Phi_y^2)]$$

Three subdomains are distinguished in the domain of the perturbed flow. The section of the piston with the specified law of motion and the characteristic A_1B_1 are the boundaries of the subdomain 0. We obtain the equation

$$\phi(x,y,t)=t-x\sin\alpha+y\cos\alpha-1=0 \tag{1.2}$$

of this characteristic from (1.1) and the condition of the adjacency of A_1B_1O to the quiescence zone, where the components of the velocity vector $u_1=\Phi_x=0$, $u_2=\Phi_y=0$. Subdomains 1 and 2 are adjacent to the planes of symmetry and subdomain 0, respectively with respect to the characteristics $\mu=m(x,y,t)=0$, $\nu=n(x,y,t)=0$ which pass through the points A and B. In the general case, these characteristics are unknown and can be determined from the system of equations for the characteristic band of Eq. (1.1) after the solution in subdomain 0 has been constructed.

The solution is constructed in the form of characteristic series: in subdomain 0 with respect to the variable ϕ, in subdomain 1 with respect to the variables μ and ϕ, and in subdomain 2 with respect to the variables ν and ϕ.

In order to construct the solution in subdomain 1 we introduce the variables

$$t_1=t,\ \ \mu=m(x,y,t),\ \ \phi(x,y,t)=t-x\sin\alpha+y\cos\alpha-1,\ \ \Psi(\mu,\phi,t_1)=\Phi(x,y,t)$$

after which Eq. (1.1) takes the form

$$\begin{aligned}
&\Psi_{tt}+\Psi_{\mu\mu}\{m_t^2-Q+(\gamma-1)Q\Psi_t+(\gamma+1)m_tQ\Psi_\mu+[2m_tP+(\gamma-1)Q]\Psi_\phi+\\
&+\tfrac{1}{2}(\gamma+1)Q^2\Psi_\mu^2+(\gamma+1)PQ\Psi_\mu\Psi_\phi+[P^2+\tfrac{1}{2}(\gamma-1)Q]\Psi_\phi^2\}+\\
&+\Psi_{\phi\phi}\{(\gamma-1)\Psi_t+[(\gamma-1)m_t+2P]\Psi_\mu+(\gamma+1)\Psi_\phi+[P^2+\tfrac{1}{2}(\gamma-1)Q]\Psi_\mu^2+\\
&+(\gamma+1)P\Psi_\mu\Psi_\phi+\tfrac{1}{2}(\gamma+1)\Psi_\phi^2\}+\\
&+\Psi_{\mu t}(2m_t+2Q\Psi_\mu+2P\Psi_\phi)+2\Psi_{\phi t}(1+P\Psi_\mu+\Psi_\phi)+\\
&+\Psi_{\mu\phi}[2(m_t-P)+2(\gamma-1)P\Psi_t+2(Q+\gamma m_tP)\Psi_\mu+2(m_t+\gamma P)\Psi_\phi+\\
&+(\gamma+1)PQ\Psi_\mu^2+2(Q+\gamma P^2)\Psi_\mu\Psi_\phi+(\gamma+1)P\Psi_\phi^2]+\\
&+\Psi_\mu(m_{tt}-W)+(\gamma-1)W\Psi_t\Psi_\mu+\Psi_\mu\Psi_\phi[2P_t+(\gamma-1)W]+\\
&+\Psi_\mu^2[W_t+(\gamma-1)m_tW]-\Psi_\mu\Psi_\phi^2[P_x\sin\alpha-P_y\cos\alpha-\tfrac{1}{2}(\gamma-1)W]+
\end{aligned}$$

$$+\ \Psi_\mu^2\Psi_\phi[(\gamma-1)WP+2m_xP_x+2m_yP_y]+$$
$$+\ \Psi_\mu^3[m_{xx}m_x^2+2m_{xy}m_xm_y+m_{yy}m_y^2+\tfrac{1}{2}(\gamma-1)WQ]=0 \tag{1.3}$$

where

$$P=-m_x\sin\alpha+m_y\cos\alpha,\quad Q=m_x^2+m_y^2,\quad W=m_{xx}+m_{yy}$$

In subdomain 2, Eq. (1.1) is written in a similar way in the variables t_1, ϕ, ν. The resulting equation and Eq. (1.3) are used to find a solution in the subdomain 0: the solution is sought both in the variables t_1, ϕ, μ and in the variables t_1, ϕ, ν, depending on which solution in which subdomain it will be matched with. The condition for matching the solutions in the subdomain 0 which have been found is then written out.

We will represent the velocity potential in subdomain 0 in the form of a series

$$\Psi(\mu,\phi,t_1)=\sum_{k=0}^{\infty}a_k(\mu,t_1)\phi^k \tag{1.4}$$

Since the perturbed gas flow along the characteristic $\phi = 0$ adjoins a quiescence zone where

$$\Phi=C=\text{const},\quad \Phi_x=\Psi_\mu m_x-\Psi_\phi\sin\alpha=0,\quad \Phi_y=\Psi_\mu m_y+\Psi_\phi\cos\alpha=0$$

then

$$\Psi_\phi(\mu,0,t_1)=\Psi_\mu(\mu,0,t_1)=0\quad\text{when}\quad m_x\cos\alpha+m_y\sin\alpha\neq 0$$

Taking account of expansion (1.4), we obtain the first coefficients of the series

$$a_0(\mu,t_1)=C=\text{const},\quad \frac{\partial a_0(\mu,t_1)}{\partial\mu}=a_1(\mu,t_1)=0 \tag{1.5}$$

We substitute expression (1.4) into (1.3) for subdomain 1, multiply the series, and equate the coefficients of like powers of ϕ to zero. As a result, we obtain first-order partial differential equations for the coefficients a_{k+1} $(k \geqslant 1)$

$$\frac{\partial a_{k+1}}{\partial t_1}+(m_t-P)\frac{\partial a_{k+1}}{\partial\mu}+(k+1)(\gamma+1)a_2a_{k+1}=F_k \tag{1.6}$$

The right-hand sides F_k of the equations depend on a, where $i \leqslant k$.

When $k = 0$. Eq. (1.6) is automatically satisfied by virtue of relations (1.5). When $k = 1$, we obtain, taking (1.5) into account

$$\frac{\partial a_2}{\partial t_1}+R\frac{\partial a_2}{\partial\mu}+(\gamma+1)a_2^2=0;\quad R=m_t+m_x\sin\alpha-m_y\cos\alpha \tag{1.7}$$

On making the change of variables

$$\mu_0=\mu,\quad t_0=\mu-Rt_1 \tag{1.8}$$

we obtain the solution of Eq. (1.7)

$$a_2(\mu_0,t_0)=\frac{R}{(\gamma+1)[\mu_0-f_2(t_0)]}=\frac{R}{(\gamma+1)T(t_0,\mu_0)} \tag{1.9}$$

which depends on the arbitrary function $f_2(t_0)$. It must be determined from the boundary condition, that is, from the impermeability condition on the section of the piston AB (Fig. 1) with a specified law of motion.

In the case of the coefficient a_{k+1} $(k \geqslant 2)$, after substituting the variables (1.8) into (1.6), we obtain the differential equation

$$R\frac{\partial a_{k+1}(\mu_0,t_0)}{\partial\mu_0}+(k+1)(\gamma+1)a_2(\mu_0,t_0)a_{k+1}(\mu_0,t_0)=\frac{F_k(\mu_0,t_0)}{2(k+1)} \tag{1.10}$$

Its solution

$$a_{k+1} = a_2^{k+1}\left(\frac{\gamma+1}{R}\right)^{k+1}\left[\frac{1}{2(k+1)(\gamma+1)^{k+1}}\int\frac{R^k F_k}{a_2^{k+1}}d\mu_0 - f_{k+1}(t_0)\right] \tag{1.11}$$

depends on the arbitrary function $f_{k+1}(t_0)$, which is also determined from the boundary condition. The recurrence formula (1.11) enables us to calculate the coefficients of series (1.4) successively. For instance, where $k = 2$, from Eq. (1.10) in the variables t_0, μ_0 and taking account of relations (1.5) and (1.9), we shall have

$$a_3 = \frac{1}{6(\gamma+1)T^3}\{\int[f_2''B_1T + 2(f_2'-1)^2B_1]d\mu_0 + (f_2'-1)\int B_2 d\mu_0\} + f_3(t_0) \tag{1.12}$$

$$B_1 = (m_x\cos\alpha + m_y\sin\alpha)^2, \quad B_2 = -6PR - (m_{tt} - m_{xx} - m_{yy})T$$

where a_3 depends on the parameters of the characteristic $\mu = m\ (t, x, y)$ and is found numerically in the general case.

The equations and their solutions when $k \geqslant 3$ are written out in a similar manner.

The solution in subdomain 1 is constructed in the form of a double series

$$\Psi(\mu,\phi,t_1) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} H_{k,j}(t_1)\mu^k\phi^j \tag{1.13}$$

From the condition for matching the solutions in subdomains 1 and 0 along the characteristic $\mu = 0$ when $a_k(0, t_1) = a_k(0, t_0)|_{\mu=0}$, we obtain the relation

$$H_{0,j}(t_1) = a_j(0,t_1) \tag{1.14}$$

At the same time, it follows from relations (1.14) and (1.15) that

$$H_{0,0}(t_1) = a_0(0,t_1) = C = \text{const}, \quad H_{0,1}(t_1) = a_1(0,t_1) = 0 \tag{1.15}$$

When $y = 0$, the condition of impermeability across the plane of symmetry

$$\Phi_y(x,0,t) = \Psi_\mu(\mu^0,\phi^0,t_1)m_y(0,x,t) + \Psi_\phi(\mu^0,\phi^0,t_1)\cos\alpha = 0 \tag{1.16}$$

must be satisfied, where

$$\phi^0 = t - x\sin\alpha - 1, \quad \mu^0 = \mu(x,0,t)$$

We now eliminate x and expand μ^0 in series in ϕ^0

$$\mu^0 = \mu\left(0, \frac{t-\phi^0-1}{\sin\alpha}, t\right) = \sum_{l=0}^{\infty} c_l(t)(\phi^0)^l \tag{1.17}$$

Since $\mu = 0$ and $\phi = 0$ intersect on the axis $y = 0$, then $c_0 = 0$.

We substitute expansions (1.13) and (1.17) into condition (1.16), equate the coefficients of $(\phi^0)^k$ to zero, and obtain the condition in the plane of summetry $y = 0$

$$m_y|_{y=0}\sum_{j=0}^{k}(k-j+1)H_{k-j+1,j}\sum_{l=0}^{\infty}C_{k-j,l} = -\cos\alpha\sum_{j=0}^{k}(j+1)H_{k-j,j+1}\sum_{l=0}^{\infty}C_{k-j,l} \tag{1.18}$$

where

$$C_{kj} = \sum_{l=0}^{j} C_{k-1,l}c_{j-l}\ (k \geqslant 3), \quad C_{2j} = \sum_{l=0}^{j} c_l c_{j-l}, \quad C_{1j} = c_j, \quad \sum_{l=0}^{\infty} C_{0,l} = 1$$

It follows from condition (1.18), when $k = 0$ and $M_y = m_y|_{y=0} \neq 0$, that

$$H_{1,0} = 0 \tag{1.19}$$

In order to find the subsequent coefficients of the series, expansion (1.13) is substituted into Eq. (1.3), the series are multiplied, and the coefficients of $\mu^k\phi^j$ are equated to zero. As a result, we obtain an equation which relates the $H_{i,j}$ $(i, j \leqslant k+2)$ (this equation is not given here because of it length).

It turned out that the groups of coefficients $H_{k,j}$ for $n = k+j$ are uniquely determined from systems of linear algebraic equations consisting of the matching conditions (1.14) when $j = n+2$, the symmetry conditions (1.18) when $k = n+1$ and $m+1$ equations for $0 \leqslant k \leqslant n$ of the form

$$(k+2)(m_t^2 - Q)H_{k+2,n-k} + 2(n-k+1)(m_t - P)H_{k+1,n-k+1} = S_{k,j} \tag{1.20}$$

where S_{kj} is an expression which depends on $H_{i,l}$ $(i, l \geqslant 0,\ i+l \leqslant n+1)$.

Thus, for $n = 0$, the system of equations, when account is taken of the coefficients (1.15) and (1.19) obtained, consists of relation (1.14) for $j = 2$, the symmetry condition (1.18) when $k = 1$

$$m_y\,|_{y=0}\,[2H_{2,0} + H_{1,1}] = -\cos\alpha[H_{1,1} + 2H_{0,2}]$$

and Eq. (1.20) for $k = 0$

$$(m_t^2 - Q)H_{2,0} = -(m_t - P)H_{1,1}$$

When $m_t^2 - Q \neq 0$, from the system of equations we find that

$$H_{1,1} = -2Y,\quad H_{2,0} = 2Y(m_t - P)(m_t^2 - Q)$$

where

$$Y = \frac{H_{0,2}\cos\alpha}{M_y[-2(m_t - P)/(m_t^2 - Q) + 1] + \cos\alpha}$$

In the self-similar case when $m_t^2 - Q = 0$ and $m_t - P \neq 0$

$$H_{1,1} = 0,\quad H_{2,0} = -H_{0,2}\cos\alpha / M_y \tag{1.21}$$

In order to determine the subsequent group of coefficients, Eqs (1.20) for $n = k+j = 1$, symmetry condition (1.18) for $k = 2$ and matching condition (1.14) for $j = 3$ are written out and account is taken of $H_{i,j}$ $(i, j = 0, 1, 2)$ which are already known. As a result, we obtain a system of four algebraic equations in the four unknown coefficients $H_{0,3}$, $H_{1,2}$, $H_{2,1}$ and $H_{3,0}$

$$\begin{aligned}
&H_{0,3}(t_1) = a_3(0, t_1)\\
&2(m_t - P)H_{1,2} + (m_t^2 - Q)H_{2,1} = F_{0,1}\\
&2(m_t - P)H_{2,1} + 3(m_t^2 - Q)H_{3,0} = F_{1,0}\\
&(M_y + 2\cos\alpha)H_{1,2} + (2M_y + \cos\alpha)H_{2,1} + 3M_yH_{3,0} = F_c
\end{aligned} \tag{1.22}$$

where

$$\begin{aligned}
F_{0,1} = &-2\frac{\partial H_{0,2}}{\partial t_1} - m_t\frac{\partial H_{1,1}}{\partial t_1} - 2(\gamma+1)H_{0,2}^2 - (\gamma+1)[m_t + 2P]H_{0,2}H_{1,1} -\\
&- 2[2m_tP + (\gamma-1)Q]H_{0,2}H_{2,0} - (\gamma+1)m_tQH_{1,1}H_{2,0} - (Q + \gamma m_tP)H_{1,1}^2 -\\
&- \tfrac{1}{2}(m_{tt} - m_{xx} - m_{yy})H_{1,1}\\
F_{1,0} = &-\frac{\partial H_{1,1}}{\partial t_1} - 2m_t\frac{\partial H_{2,0}}{\partial t_1} - (\gamma+1)H_{0,2}H_{1,1} -\\
&- 2[(\gamma-1)m_t + 2P]H_{0,2}H_{2,0} - (m_t + \gamma P)H_{1,1}^2 -\\
&- (\gamma+1)[2m_tP + Q]H_{1,1}H_{2,0} - 2(\gamma+1)m_tQH_{2,0}^2 - (m_{tt} - m_{xx} - m_{yy})H_{2,0}\\
F_c = &-3\cos\alpha H_{0,3}
\end{aligned}$$

Solving this system of equations, we find $H_{i,j}$ for $i + j = 3$.

Next, the systems of equations for the coefficients $H_{k,j}$ $(k + j = i + 1)$ are successively written out for $n = k + j$, $n = 1, 2, \ldots$. These coefficients depend on a_l $(l \leqslant i + 1)$ which are determined by the specified boundary condition, that is, by the law of motion of the piston in subdomain 0.

In order to construct a solution in subdomain 2, we introduce the variables

$$t_1 = t,\ \ \nu = n(x,y,t),\ \ \phi(x,y,t) = t - x\sin\alpha + y\cos\alpha - 1,\ \ \Lambda(\nu,\phi,t_1) = \Phi(x,y,t)$$

Equation (1.1) in this subdomain will have the form (1.3) but here it is necessary to replace μ by ν, m by n, and Ψ by Λ.

In a similar manner to the construction of the solution in subdomain 1, the velocity potential is sought in the form of the double series

$$\Lambda(\nu,\phi,t_1) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} G_{k,j}(t_1)\nu^k\phi^j \tag{1.23}$$

At the same time, the solution in subdomain 0 is reconstructed in the form of the series

$$\Lambda(\nu,\phi,t_1) = \sum_{k=0}^{\infty} b_k(\nu,t_1)\phi^k \tag{1.24}$$

with coefficients $b_k(\nu, t_1)$ which depend on the characteristic variable ν and satisfy equations and relations (1.5)–(1.11) with a, μ and m replaced in them by b, ν and n. The equality

$$a_k(\mu,t_1) = b_k(\nu,t_1) \tag{1.25}$$

follows from relations (1.4) and (1.24) for the resulting solutions to be compatible in subdomain 0.

The symmetry condition

$$\Phi_x(0,y,t) = \Lambda_\nu n_x - \Lambda_\phi \sin\alpha = 0$$

must be satisfied in subdomain 2. After eliminating y, expansion in series

$$\nu^0 = \nu\left(\frac{\phi^0 - t + 1}{\cos\alpha}, 0, t\right) = \sum_{l=0}^{\infty} d_l(t)(\phi^0)^l$$

and substitution of series (1.23), this symmetry condition will have the form

$$n_x|_{x=0}\sum_{j=0}^{k}(k-j+1)G_{k-j+1,j}\sum_{l=0}^{\infty}D_{k-j,l} = \sin\alpha\sum_{j=0}^{k}(j+1)G_{k-j,j+1}\sum_{l=0}^{\infty}D_{k-j,l}$$

where

$$D_{kj} = \sum_{l=0}^{\infty} d_{j-l}D_{k-1,l},\ (k \geqslant 3),\ \ D_{2j} = \sum_{l=0}^{j} d_l d_{j-l},\ \ D_{1j} = d_j,\ \ \sum_{l=0}^{\infty} D_{0,l} = 1$$

The equations and relations for G_k will be analogous to the equations and relations for H_k, but taking the second symmetry condition into account.

2. EXAMPLE OF A CALCULATION

As an example and test, we will consider the shock-free compression of a prism according to a specified law of motion of the piston CD in the segment AB (Fig. 1), when the gas flow is self-similar in subdomains 0 and 1.

The problem of the adiabatic compression at constant entropy of a gas prism with a section OSK and an angle α, which is matched with the adiabatic exponent γ and satisfies the relations

$$\sin\alpha = \sqrt{3-\gamma}/2,\ \ \cos\alpha = \sqrt{\gamma+1}/2$$

has been considered earlier in [1–3].

The exact expression for the potential in subdomain 0 in the characteristic variables has the form [2]

$$\Phi = \frac{1}{\gamma - 1} + \frac{1}{\gamma + 1}\frac{\phi^2}{\tau}, \quad \tau = \frac{t}{t_*} - 1 < 0 \tag{2.1}$$

t_* is the time taken by the piston to traverse the distance KO. Following the well-known procedure [1], the exact solution in subdomain 1

$$\Phi = \frac{1}{\gamma - 1} + \frac{\phi^2 + \mu^2}{(\gamma + 1)\tau} \tag{2.2}$$

is also written out. The equation of the curvilinear part CA of the piston CF

$$\left[x + \frac{\sqrt{3-\gamma}}{\gamma - 1}\tau\right]\left[-y + \frac{\sqrt{\gamma+1}}{\gamma - 1}\tau\right] = \frac{\sqrt{(\gamma+1)^3}}{(\gamma-1)^2\sqrt{3-\gamma}}(-\tau)^{4/(\gamma+1)} \tag{2.3}$$

has been obtained in [3], after which it is easy also to write out the equation of its plane part AF

$$\frac{\sqrt{3-\gamma}}{2}x - \frac{\sqrt{\gamma+1}}{2}y + \frac{\gamma+1}{\gamma-1}(-\tau)^{2/(\gamma+1)} + \frac{2}{\gamma-1}\tau = 0 \tag{2.4}$$

By a certain instant of time t_n, the piston traverses a path, equal to KF, along the wall KO and occupies the position CAF. For the test, this position of the piston is taken as the initial position. In the case of the complete problem. $CAFBD$ will correspond to the initial position of the piston.

When using the self-similar solution as a test, when the piston, at $t = 0$, occupies the position SK and $|KO| = 1$, the transition to dimensionless quantities must be carried out such that the position of the piston CAF corresponds to the time $t_n = 0$ and the linear dimensions of the prism correspond to $|OH| = 1$. In this case, the relation between the old variables x, y, τ and the new dimensionless variables x_n, y_n, τ_n will be as follows:

$$x = (1 - l_0 / L_0)x_n, \quad y = (1 - l_0 / L_0)y_n, \quad \tau = (1 - t_n / t_*)\tau_n$$

where L_0 is the length OK and l_0 is the length KN. After the introduction of the new variables, Eqs (2.1)–(2.4) retain their previous form but they will allow for the shift in time.

We shall assume that the law of motion of the piston (2.4) is specified in the segment AB. In the characteristic variables (1.2), the impermeability condition

$$U = Z_t + Z_x u_1 + Z_y u_2 = 0 \tag{2.5}$$

on the piston (2.4)

$$Z = \phi(x, y, \tau) - \Pi(\tau), \quad \Pi(\tau) = \frac{\gamma+1}{\gamma-1}[\tau + (-\tau)^{2/(\gamma+1)}] \tag{2.6}$$

takes the form

$$U(t, \mu, \phi) = \Psi_\phi - \frac{\gamma-1}{2}\Psi_\mu + 1 - \Pi'(\tau) = 0$$

Substituting Ψ in the form of series (1.4), we obtain

$$U(t, \mu, \Pi(\tau)) = \sum_{k=0}^{\infty}\left[(k+1)a_{k+1}(\mu, t) - \frac{\gamma-1}{2}\frac{\partial a_k(\mu, t)}{\partial \mu}\right]\Pi^k(\tau) -$$
$$- \Pi'(\tau) + 1 = 0 \tag{2.7}$$

at the same time keeping in mind relations (1.5).

When $t = 0$, equality (2.7) is automatically satisfied.

To find a_2 we put $t = 0$ in the equality obtained by differentiating relation (2.7) with respect to t

$$\sum_{k=0}^{\infty}\left[(k+1)\frac{\partial a_{k+1}(\mu,t)}{\partial t}-\frac{\gamma-1}{2}\frac{\partial^2 a_k(\mu,t)}{\partial t\partial\mu}\right]\Pi^k(\tau)+$$
$$+\sum_{k=0}^{\infty}\left[(k+2)a_{k+2}(\mu,t)-\frac{\gamma-1}{2}\frac{\partial a_{k+1}(\mu,t)}{\partial\mu}\right](k+1)\Pi^k(\tau)\Pi'(\tau)-$$
$$-\Pi''(\tau)=0 \qquad (2.8)$$

and, taking into account a_0 and a_1 obtained, we have

$$a_2(\mu,0)=\frac{1}{2}\Pi''(-1)=-\frac{1}{\gamma+1}=C_2=\text{const} \qquad (2.9)$$

It follows from relations (1.9) that

$$a_2(\mu,t_0)=\frac{1}{2[\mu-f_2(t_0)]},\quad t_0=\mu-\frac{\gamma+1}{2}t$$

and, from equality (2.9), we have

$$a_2(\mu,t)\,|_{t=0}=a_2(\mu,t_0)\,|_{t=0}=a_2(\mu,\mu)=\frac{1}{2[\mu-f_2(\mu)]}=C_2=-\frac{1}{\gamma+1}=\text{const}$$

Hence,

$$f_2(\mu)=\mu+\frac{\gamma+1}{2},\quad a_2(\mu,t)=a_2(\mu_0,t_0)=\frac{1}{2[\mu_0-f_2(t_0)]}=\frac{1}{(\gamma+1)(t-1)} \qquad (2.10)$$

Differentiating equality (2.8) with respect to t and putting $t = 0$, we obtain $a_3(\mu, 0) = 0$ and, from relation (1.12) when account is taken of the first expression of (2.10), we have

$$a_3(\mu_0,t_0)=-f_3(t_0)/[\mu_0-f_2(t_0)]^3$$

Similar procedures as when finding f_2, a_2, give

$$f_3(\mu)=0,\quad a_3(\mu,t)=0 \qquad (2.11)$$

In calculating the subsequent coefficients a_l of the series, each time the next derivative of the impermeability condition is differentiated with respect to t, t is put equal to 0 and the coefficients a_k $(k < l)$ which have already been found are taken into account. As a result, we find

$$a_l(\mu,t)=0,\quad l>3 \qquad (2.12)$$

Substituting the a_k obtained into series (1.4), we obtain the solution in subdomain 0

$$\Psi(\mu,\phi,t)=a_0+a_2\phi^2+\ldots=a_0+\frac{1}{(\gamma+1)(t-1)}\phi^2$$

which is identical to the exact solution when $a_0 = C = 1/(\gamma-1)$.

A knowledge of the equations of the characteristics $\mu(x, y, \tau) = 0$, $\nu(x, y, \tau) = 0$, along which the required solution will adjoin the solution which has been found in subdomain 0, is required in order to construct the solutions in subdomains 1 and 2.

We write out the system of equations for the characteristic band [7], putting $q(x, y, \tau) = x - X(y, \tau) = 0$, where q is any of the characteristic variables μ, ν, ϕ. We now eliminate c^2 and obtain the equation

$$\frac{dX_y}{d\tau}=\frac{1}{(\gamma+1)\tau}\left[\frac{\sqrt{3-\gamma}}{2}X_y-\frac{\sqrt{\gamma+1}}{2}\right]\left[\sqrt{3-\gamma}+\sqrt{\gamma+1}X_y\pm(\gamma-1)\sqrt{1+X_y^2}\right]$$

We shall assume that the expression in the first set of square brackets vanishes. It has been found that this case corresponds to the characteristic $\phi(x, y, \tau) = 0$ (1.2). From the fact that the expression in the second set of square brackets with the lower (minus) sign is equal to zero, we obtain the same characteristic.

We obtain the characteristic

$$\mu(x,y,\tau)=\tau-\frac{\sqrt{3-\gamma}}{2}x-\frac{\sqrt{\gamma+1}}{2}y=0 \tag{2.13}$$

from the condition that the second expression with the upper (plus) sign vanishes, from the solution of the other equations of the band and from the condition for it to intersect the characteristic $\phi = 0$ on the $y = 0$ axis. If the condition for it to intersect the characteristic $\phi = 0$ on the $x = 0$ axis is considered, we obtain the characteristic

$$\nu(x,y,\tau)=-\tau-\frac{\sqrt{3-\gamma}}{2}x-\frac{\sqrt{\gamma+1}}{2}y=0 \tag{2.14}$$

We will assume that both expressions do not vanish, and, then, the system of equations for the characteristic band reduces, after some reduction, to the relation

$$F(X_y)=\left[\frac{\sqrt{\gamma+1}-\sqrt{3-\gamma}X_y}{-\sqrt{3-\gamma}-\sqrt{\gamma+1}X_y+(\gamma-1)\sqrt{1+X_y^2}}\frac{2(\gamma-1)}{\sqrt{\gamma+1}\sqrt{3-\gamma}}\right]-$$
$$-\left(\frac{\zeta^-(X_y)}{\zeta^+(X_y)}\frac{\sqrt{3-\gamma}+2}{\sqrt{3-\gamma}}\right)^{(\gamma-1)/2}=0$$

where

$$\zeta^\pm(X_y)=\sqrt{\gamma+1}\left(1+\sqrt{1+X_y^2}\right)-(\sqrt{3-\gamma}\pm 2)X_y$$

and the two equations

$$\frac{dy}{d\tau}=\left(\frac{1}{\sqrt{\gamma+1}}+\frac{\gamma-1}{\gamma+1}\frac{X_y}{\sqrt{1+X_y^2}}\right)\varphi-\frac{X_y}{\sqrt{1+X_y^2}}$$
$$\frac{dX}{d\tau}=-\frac{\sqrt{3-\gamma}}{\gamma+1}\varphi+\left(1-\frac{\gamma-1}{\gamma+1}\varphi\right)\frac{1}{\sqrt{1+X_y^2}}$$

where

$$\varphi=1+\frac{\sqrt{\gamma+1}}{2\tau}y-\frac{\sqrt{3-\gamma}}{2\tau}X(y,\tau)$$

The functions $F(X_y)$ and X_y are expanded in Taylor series along the characteristics in the neighbourhood of the point B_1, where $q = \phi_0$. It was found that, at the point B_1

$$F=0,\quad F'_{X_y}\neq 0\quad (X_y)'_\nu=0$$

by virtue of which the function $X(y, \tau)$ is exactly defined

$$X_y(y,\tau)=-\sqrt{\frac{\gamma+1}{3-\gamma}},\quad X(y,\tau)=-\frac{2}{\sqrt{3-\gamma}}\left(\tau+\frac{\sqrt{\gamma+1}}{2}y\right)$$

and corresponds to the characteristic $\nu = 0$ (2.13).

In subdomain 1, the velocity potential is sought in the form of the double series (1.13). From relations (1.15), (1.5) and (1.19), it follows that

$$H_{0,0} = a_0(0,t) = C = \text{const}, \quad H_{0,1} = H_{1,0} = 0$$

From relation (1.14) when $j = 2$, the second expression of (2.1) and relations (1.21) and (2.11), we obtain

$$H_{2,0}(t) = H_{0,2}(t) = \frac{1}{(\gamma+1)(t-1)}, \quad H_{1,1} = 0, \quad H_{0,3}(t) = a_3(0,t) = 0$$

When $m_t^2 - Q = 0$, the system of equations (1.22) for the coefficients $H_{1,2}, H_{2,1}$ and $H_{3,0}$ will have the form

$$(1-P)H_{1,2} = -\frac{\partial H_{0,2}}{\partial t} - (\gamma+1)H_{0,2}^2 - (\gamma - 1 + 2P)H_{2,0}^2$$

$$(1-P)H_{2,1} = -\frac{\partial H_{2,0}}{\partial t} - (\gamma - 1 + 2P)H_{0,2}^2 - (\gamma+1)H_{2,0}^2$$

$$(M_y + 2\cos\alpha)H_{1,2} + (2M_y + \cos\alpha)H_{2,1} + 3M_y H_{3,0} = -3\cos\alpha H_{0,3}$$

whence, when account is taken of the equality $H_{0,2} = H_{2,0}$, we obtain $H_{1,2} = H_{2,1} = 0$. From the third equation when $M_y = -\cos\alpha = -\sqrt{\gamma + 1 / 2}$, it follows that

$$H_{0,3}(t) = H_{3,0}(t) = 0$$

From relations (1.14) and (2.12), we have

$$H_{0,4}(t) = a_4(0,t) = 0$$

We now write out the system of equations for the coefficients H_{il}, $i + l = 4$ when $k, j = 0,2;\ 1,1;\ 2,0$ and obtain

$$6RH_{1,3} + 4(\gamma+1)H_{0,2}^3 + 8\left[P^2 + \frac{1}{2}(\gamma-1)\right]H_{2,0}^3 +$$

$$+\ 2(\gamma+3)H_{0,2}\frac{\partial H_{0,2}}{\partial t_1} + 2(\gamma-1)H_{2,0}\frac{\partial H_{0,2}}{\partial t_1} + \frac{\partial^2 H_{0,2}}{\partial t_1^2} = 0$$

$$RH_{2,2} + 2(\gamma+1)PH_{0,2}^3 + PH_{2,0}\frac{\partial H_{0,2}}{\partial t_1} + PH_{0,2}\frac{\partial H_{2,0}}{\partial t_1} = 0$$

$$6RH_{3,1} + 8\left[P^2 + \frac{1}{2}(\gamma-1)\right]H_{0,2}^3 + 4(\gamma+1)H_{2,0}^3 +$$

$$+\ 2(\gamma-1)H_{0,2}\frac{\partial H_{2,0}}{\partial t_1} + 2(\gamma+3)H_{2,0}\frac{\partial H_{2,0}}{\partial t_1} + \frac{\partial^2 H_{2,0}}{\partial t_1^2} = 0$$

It follows from this that $H_{1,3} = H_{2,2} = H_{3,1} = 0$ and, from the symmetry condition, it follows that $H_{4,0} = 0$.

On continuing the calculation of the subsequent groups of coefficients, we obtain that all the coefficients vanish. As a result, in subdomain 1 the velocity potential, constructed using the specified law of motion of the plane part of the piston, is identical to the exact solution (2.2).

In subdomain 1, we now find the law of motion $F(x, y, t) = 0$ of the curvilinear part CA of the piston CD. The impermeability condition on it leads to the first order, linear, homogeneous, partial differential equation (2.5) for the unknown function F of three independent variables. The characteristics covering the surface $F = 0$ satisfy the equations

$$dx = u_1 d\tau, \quad dy = u_2 d\tau$$

or

$$dx = [(3-\gamma)x - 2\sqrt{3-\gamma}\tau]H_{0,2}(\tau)d\tau, \quad dy = (\gamma+1)yH_{0,2}(\tau)d\tau$$

On combining these equations, we obtain the total differentials, the integrals of which have the form

$$c_1 = \frac{y}{\tau}, \quad c_2 = \tau^{2\varkappa-1}x + \xi\tau^{2\varkappa}; \quad \varkappa = \frac{\gamma-1}{\gamma+1}, \quad \xi = \frac{\sqrt{3-\gamma}}{\gamma-1}$$

where c_1 and c_2 are arbitrary constants. We relate c_1 and c_2 by the functional relation $c_2 = \omega(c_1)$ and find ω from the condition that the curve passes through the point A, that is, the point of intersection of the part AB of the piston, where the law of motion is known, with the characteristic $\mu = 0$,

$$x_A = -[\xi\tau + \eta(-\tau)^{1-\varkappa}], \quad y_A = \sigma[\tau + (-\tau)^{1-\varkappa}]$$

$$\eta = \frac{\gamma+1}{(\gamma-1)\sqrt{3-\gamma}}, \quad \sigma = \frac{\sqrt{\gamma+1}}{\gamma-1}$$

The relation between c_1 and c_2 will have the form

$$c_2(1 - c_1/\sigma) = \eta$$

Substituting c_1 and c_2 from the integrals, we obtain the equation of the piston in subdomain 1, which is identical with (2.3).

Carrying out the calculations in subdomain 2 analogous to the calculations in subdomain 1, but taking into account the corresponding symmetry condition

$$-\sum_{j=0}^{k}(-1)^{k-j}(k-j+1)G_{k-j+1,j} = \sum_{j=0}^{k}(-1)^{k-j}(j+1)G_{k-j,j+1}$$

and, also, the compatibility conditions (1.25) and matching the solution with the solution in subdomain 0, we obtain the velocity potential in this subdomain

$$\Psi(t,\phi,\nu) = C + \frac{1}{(\gamma+1)(t-1)}(\phi^2 + \nu^2)$$

On taking the impermeability condition into account, we find the equation of the piston

$$2\varkappa + 1 - 2V + 2\ln(2V) = \varkappa\left[\sqrt{\gamma+1}\frac{y}{\tau} - 2\ln(-\tau)\right]$$

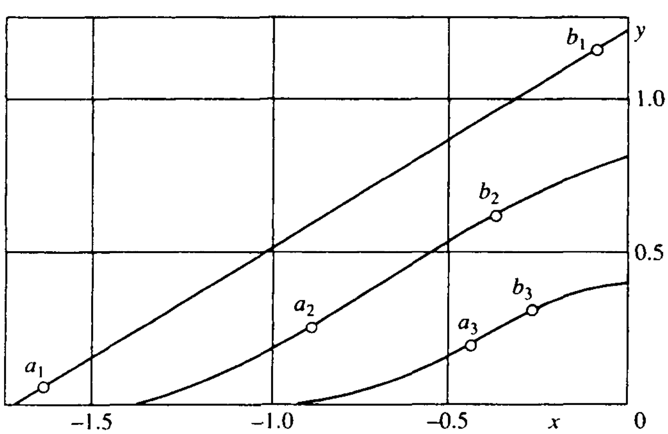

Fig. 2

where

$$V = \left[1 + \sqrt{1 + \frac{4}{\eta}(-\tau)^{2\varkappa-1}x}\right]^{-1}$$

The shape of the piston, calculated according to the specified law of its motion in the segment a_ib_i, taken from the exact solution for the instants of time $t_i = 0.1, 0.7, 0.9$, is shown in Fig. 2.

Thus, in the case of the shock-free symmetric compression of prisms, it is only necessary to specify the law of motion of the position on a certain part of it. In the segments belonging to the planes of symmetry, their law of motion is uniquely defined by the law of motion of the central part of the piston and the impermeability condition in the planes of symmetry. A new exact expression is obtained for the velocity potential and the shape of the wall of the gas prism with two planes of symmetry which is being compressed for the case when the gas is ideal and the angle α is connected with the adiabatic exponent γ by a specific relation.

The algorithm proposed can be used to calculate the initial data when computing the parameters of a gas being compressed in a symmetric prism using any numerical method.

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